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TRANSLATION NO. 568

DATE: July 1968

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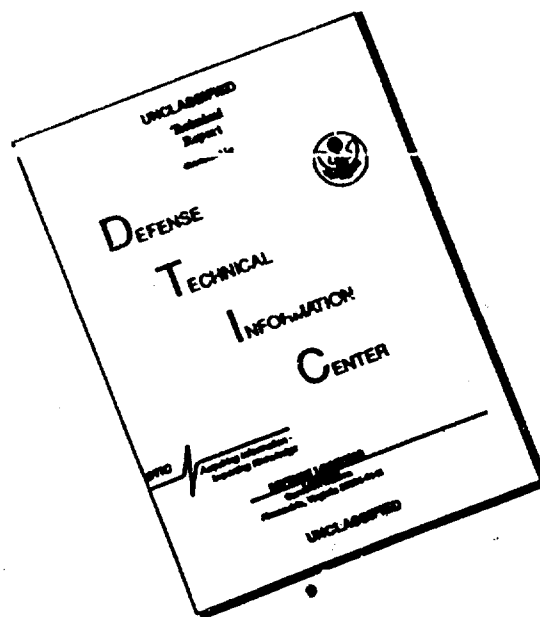
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Morris D. Friedman

Russian Translation

On Some Coupled Integral Equations

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Doklady, A.N. USSR, vol. 90, No. 3, 1954, pp. 333-336

1. Many mixed boundary problems of mathematical physics reduce to the finding of unknown functions, if the result of applying one integral operator to it is known in one part of a certain interval and the result of the application of a second integral operator is known in the remainder of the specified interval. In these cases it is said that the problem reduces to coupled integral equations. As a simple example, there is the pair of integral equations

$$\begin{aligned} F(x) &= \sqrt{\frac{x}{\pi}} \int_0^{\infty} S(\lambda) \sin \lambda x \, d\lambda \\ G(x) &= \sqrt{\frac{x}{\pi}} \int_0^{\infty} S(\lambda) \cos \lambda x \, d\lambda \end{aligned} \quad (1)$$

the first of which should be satisfied on some given system of intervals of the  $x \geq 0$  semi-axis, and the second on the rest of this semi-axis. To this trigonometric coupled equation reduce certain mixed boundary problems of the theory of the logarithmic potential for a half-plane and one of the methods of solving them is to use inversion formulas for singular integrals. If we turn from the half-plane to the half-space, then if there is an axis of symmetry we obtain coupled integral equations containing cylinder functions. Certain types of such paired equations were already considered [1,2]. The present note is devoted to two subsequent types. Since we do not speak here of the construction

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of any general theory but about specific coupled equations, then solution techniques are of the greatest interest. Consequently, we are limited in the present note to explaining formal relations and will not analyze the conditions of their applicability.

The apparatus used in our construction is due entirely to H. Ya. Sonine and is composed of the following:

- 1) The discontinuous Sonine integral (see [3], p. 64 and [4], p. 455)

$$\int_0^{\infty} J_{\mu}(bx) x^{\mu+1} \frac{J_{\nu}(a\sqrt{x^2+z^2})}{(\sqrt{x^2+z^2})^{\nu}} dx = \begin{cases} \frac{b^{\mu}}{a^{\nu}} \left( \frac{\sqrt{a^2-b^2}}{z} \right)^{\nu-\mu+1} J_{\nu-\mu+1}(z\sqrt{a^2-b^2}), & a > b > 0 \\ 0 & 0 < a < b \end{cases}$$

where  $\nu > \mu > -1$  and  $z$  is an arbitrary complex number

- 2) The Sonine inversion formula (see [3], p. 151)

$$G(x) = \int_a^x F(t) \left( \frac{\sqrt{x^2-t^2}}{1/k} \right)^{-p} J_{-p}(ik\sqrt{x^2-t^2}) dt$$

$$F(x) = \frac{d}{dx} \int_a^x G(t) t \left( \frac{\sqrt{x^2-t^2}}{k} \right)^{-q} J_{-q}(k\sqrt{x^2-t^2}) dt$$

where  $p > 0$ ,  $q > 0$ ,  $p + q = 1$ , and  $k$  is any complex number (for  $k = 0$ , the formulas transform into a generalized Abel formula).

2. Let us consider the coupled equations

$$(2) \quad \int_0^{\infty} S(\lambda) J_0(\lambda x) \lambda d\lambda = f(x) \quad (0 < x < 1)$$

$$(3) \quad \int_0^{\infty} S(\lambda) J_0(\lambda x) (\lambda^2 - k^2)^p \lambda d\lambda = 0 \quad (x > 1)$$

where  $k \geq 0$ ,  $p$  ( $0 < p^2 < 1$ ) and also  $f(x)$  ( $0 < x < 1$ ) are

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given. It is required to find the function  $S(\lambda)$  as well as

$$(4) \quad s(x) = \int_0^{\infty} S(\lambda) J_0(\lambda x) (\lambda^2 - k^2)^p \lambda d\lambda \quad (0 < x < 1)$$

Let us note that for  $k = 0$  these equations were already considered and their formal solution is known [1, 2]. Let us also note that we took the Bessel function of zero order, and not of any order  $\nu > -1$ , as the kernel only for simplicity.

Let us turn to the solution of the considered equations. First of all we put

$$(5) \quad S(\lambda) = \int_0^1 h(t) (\sqrt{\lambda^2 - k^2})^{-p} J_{-p}(t\sqrt{\lambda^2 - k^2}) dt$$

where  $h(t)$  is the new unknown function. The justification for its introduction is that for any choice of  $h(t)$  the function (5) satisfies (3), as may be confirmed by using the Sonine discontinuous integral.

To determine  $h(t)$  there remains now to substitute (5) into (2), which can be done if  $p > 0$ . If  $p < 0$ , then we take in place of (2) the equation which results from it

$$(2 \text{ bis}) \quad \frac{1}{x} \int_0^x f(\xi) \xi d\xi = \int_k^{\infty} S(\lambda) J_1(\lambda x) d\lambda$$

Making this substitution and again using the Sonine discontinuous integral, we find that

$$f(x) = \int_0^x h(t) t^{-p} \left( \frac{\sqrt{x^2 - t^2}}{k} \right)^{p-1} J_{p-1}(k\sqrt{x^2 - t^2}) dt$$

if  $p > 0$ , and

$$f(x) = \frac{1}{x} \frac{d}{dx} \int_0^x h(t) t^{-p} \left( \frac{\sqrt{x^2 - t^2}}{k} \right)^p J_p(k\sqrt{x^2 - t^2}) dt$$

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if  $p < 0$ . Hence, with the aid of the Sonine inversion formula we obtain the following expression for the auxiliary function  $h(t)$ :

$$h(t) = t^p \frac{d}{dt} \int_0^t f(x) x \left( \frac{\sqrt{t^2 - x^2}}{1k} \right)^{-p} J_{-p}(1k\sqrt{t^2 - x^2}) dx$$

if  $p > 0$ , and

$$h(t) = t^{1+p} \int_0^t f(x) x \left( \frac{\sqrt{t^2 - x^2}}{1k} \right)^{-1-p} J_{-1-p}(1k\sqrt{t^2 - x^2}) dx$$

if  $p < 0$ .

To determine  $g(x)$ , equation (4) is used or the equality resulting from (3) and (4) (with the aid of the Hankel inversion formula):

$$S(\lambda)(\lambda^2 - k^2)^p = \int_0^1 \xi g(\xi) J_0(\lambda\xi) d\xi$$

By this means we arrive at the formulas:

$$S(x) = \frac{1}{x} \frac{d}{dx} \int_x^1 h(t) t^p \left( \frac{\sqrt{t^2 - x^2}}{1k} \right)^{-p} J_{-p}(1k\sqrt{t^2 - x^2}) dt$$

if  $p > 0$ , and

$$S(x) = \int_x^1 h(t) t^{1+p} \left( \frac{\sqrt{t^2 - x^2}}{1k} \right)^{-1-p} J_{-1-p}(1k\sqrt{t^2 - x^2}) dt$$

if  $p < 0$ :

3. The following coupled integral equations can be solved by an analogous method:

$$(6) \quad \begin{aligned} \int_k^\infty g(\lambda) J_p(\lambda x) d\lambda &= f(x) & (0 < x < 1) \\ \int_0^\infty g(\lambda) J_{-p}(\lambda x) d\lambda &= 0 & (x = 1) \end{aligned}$$

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where the number  $k \geq 0$ , the number  $\nu$  ( $0 < \nu^2 < 1$ ) and the function  $f(x)$  ( $0 < x < 1$ ) are given; and  $c(\lambda)$  is to be found as well as

$$c(x) = \int_0^{\infty} c(\lambda) J_{-\nu}(\lambda x) d\lambda \quad (0 < x < 1)$$

Let us note that for  $\nu = 1/2$ , (5) becomes

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} c(\lambda) \sin \lambda x \frac{d\lambda}{\sqrt{\lambda}} = \sqrt{x} f(x) \quad (0 < x < 1)$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} c(\lambda) \cos \lambda x \frac{d\lambda}{\sqrt{\lambda}} = 0 \quad (x > 1)$$

and, if  $k > 0$ , is not a particular case of (1).

The solution of (5) for  $\nu > 0$  is given by

$$c(\lambda) = \lambda^{1-\nu} \int_0^1 h(t) J_0(t\sqrt{\lambda^2 - k^2}) dt$$

$$c(x) = x^{-\nu} \int_0^1 h(t) \left( \frac{\sqrt{t^2 - x^2}}{ik} \right)^{\nu-1} J_{\nu-1}(ik\sqrt{t^2 - x^2}) dt$$

where the auxiliary function  $h(t)$  is determined by the equality

$$h(t) = \frac{d}{dt} \int_0^t f(x) x^{1+\nu} \left( \frac{\sqrt{t^2 - x^2}}{ik} \right)^{-\nu} J_{-\nu}(ik\sqrt{t^2 - x^2}) dx$$

We do not dwell on the case  $\nu < 0$  since it reduces to that already considered, if, first of all, we replace (6) by

$$\int_k^{\infty} c(\lambda) J_{1+\nu}(\lambda x) \lambda d\lambda = -x^{\nu} \frac{d}{dx} [x^{-\nu} f(x)] \quad (0 < x < 1)$$

$$\int_0^{\infty} c(\lambda) J_{-1-\nu}(\lambda x) \lambda d\lambda = 0 \quad (x > 1)$$

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June, 1954

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